Sequences and Limits

Definition (c.f. Definition 3.1.1)**.** A *sequence (of real numbers)* is a function $X : \mathbb{N} \to \mathbb{R}$. The n-th term of the sequence is denoted by $x_n = X(n)$ and we usually write

$$X = (x_n) \quad \text{or} \quad X = (x_1, x_2, x_3, ...).$$

Remark. There are always an infinite number of terms in a sequence because the domain of the function is $\mathbb{N}$. There are **no** "sequences" like $(1, 2, 3)$ or $(0, 0, 0, ..., 0, 1)$. Different from a set, the order of the terms of a sequence matters and repetition of terms is allowed. For example, the sequences $(1, 2, 3, 1, 2, 3, ...)$ and $(3, 2, 1, 3, 2, 1, ...)$ are not the same. However, the sets $\{1, 2, 3, 1, 2, 3, ...\}$ and $\{3, 2, 1, 3, 2, 1, ...\}$ both represent the finite set $\{1, 2, 3\}$.

Example 1 (c.f. Example 3.1.2)**.** Consider the following example of sequences.

- Let $b \in \mathbb{R}$. The sequence $B = (b, b, b, ...)$ is called a *constant sequence*, every term in the sequence are equal.
- The sequence $X = (10^{-n})$ represents the sequence $(0.1, 0.01, 0.001, 0.0001, ...)$. In this case, the terms of the sequence X is given by a **formula**.
- The terms of the *Fibonacci sequence* $F = (f_n) = (1, 1, 2, 3, 5, 8, ...)$ can be given by the following **inductive formula**:

  $$f_1 = 1, \quad f_2 = 1, \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \quad \text{for } n \geq 3.$$

  Remark. f_n can also be given by an explict formula:

  $$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

Definition (c.f. Definition 3.1.3)**.** Let (x_n) be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. (x_n) is said to *converge to* x if for every $\varepsilon > 0$, there exist a natural number $N \in \mathbb{N}$ such that

$$|x_n - x| < \varepsilon, \quad \forall n \geq N.$$

In this case, x is said to be the *limit* of (x_n) and is denoted by $x = \lim(x_n)$. A sequence is said to be *convergent* if it has a limit and *divergent* if it is not convergent.

Remark. Notice that:

- In the definition, the number $x \in \mathbb{R}$ is first specified and then proven to be the limit. In other words, we have to make a "guess" of the limit of the sequence first.
- The limit of a sequence is unique (c.f. 3.1.4 Uniqueness of Limits). i.e., If x and y are both a limit of a sequence (x_n), then $x = y$. Thus, we denote $x = \lim x_n$.
- If a sequence (x_n) is divergent, it cannot converge to **any** real number $x \in \mathbb{R}$. i.e., for any $x \in \mathbb{R}$, there exists an $\varepsilon > 0$ such that for any $N \in \mathbb{N}$, there exists some $n \geq N$ such that $|x_n - x| \geq \varepsilon$.

Example 2 (c.f. Example 3.1.6(d))**.** Show that $\lim(\sqrt{n+1}-\sqrt{n})=0$.

Solution. We need to show that $\forall\varepsilon>0$, $\exists N\in\mathbb{N}$ such that

$$|(\sqrt{n+1}-\sqrt{n})-0|<\varepsilon,\quad \forall n\geq N.$$

Let simplify the absolute value that we need to estimate:

$$|(\sqrt{n+1}-\sqrt{n})-0|=\sqrt{n+1}-\sqrt{n}=\frac{1}{\sqrt{n+1}+\sqrt{n}}\leq\frac{1}{\sqrt{n}}$$

We need to pick some natural number N with the assumption $n\geq N$, then

$$|(\sqrt{n+1}-\sqrt{n})-0|\leq\frac{1}{\sqrt{n}}\leq\frac{1}{\sqrt{N}}.$$

Thus for any given $\varepsilon>0$, we pick $N\in\mathbb{N}$ such that $1/\sqrt{N}<\varepsilon$. It can be achieved by applying the **Archimedean Property** on the number $\varepsilon^2>0$.

Proof. Let $\varepsilon>0$. By **Archimedean Property**, there exists $N\in\mathbb{N}$ such that

$$\frac{1}{N}<\varepsilon^2 \iff \frac{1}{\sqrt{N}}<\varepsilon$$

Hence whenever $n\geq N$,

$$|(\sqrt{n+1}-\sqrt{n})-0|=\frac{1}{\sqrt{n+1}+\sqrt{n}}\leq\frac{1}{\sqrt{n}}\leq\frac{1}{\sqrt{N}}<\varepsilon.$$

The result follows. □

Exercise. Show that $\lim(1/n)=0$.

Example 3. Show that $\lim(\sqrt{n+1}-\sqrt{n})\neq 1$.

Solution. We need to show that $\exists\varepsilon>0$ such that $\forall N\in\mathbb{N}$, there exists $n\geq N$ such that

$$|(\sqrt{n+1}-\sqrt{n})-1|\geq\varepsilon.$$

Let simplify the absolute value that we need to estimate:

$$|(\sqrt{n+1}-\sqrt{n})-1|=1-(\sqrt{n+1}-\sqrt{n})\geq 1-\frac{1}{\sqrt{n}}$$

Notice that if $n\geq 2$, the estimate always greater than or equal to $1-1/\sqrt{2}\approx 0.2929$. Thus we can pick $\varepsilon=0.1$.

Proof. Take $\varepsilon=0.1$. Then whenever $N\in\mathbb{N}$, choose $n=\max\{N,2\}$. Hence $n\geq N$ and

$$|(\sqrt{n+1}-\sqrt{n})-1|\geq 1-\frac{1}{\sqrt{n}}\geq 1-\frac{1}{\sqrt{2}}\geq 0.1=\varepsilon.$$

The result follows. □

Exercise. Show that $\lim(1/n)\neq 100$.

Example 4. Find the limit of the sequence $\left(\dfrac{4n-3}{2n-7}\right)$ and prove your assertion.

Solution. From secondary school calculus, we have

$$\lim_{n\to\infty}\frac{4n-3}{2n-7}=\lim_{n\to\infty}\frac{4-3/n}{2-7/n}=\frac{4-0}{2-0}=2.$$

Let's prove this by definition. Let $\varepsilon>0$. Note that if $n\geq N\geq 4$, we have

$$\left|\frac{4n-3}{2n-7}-2\right|=\frac{11}{2n-7}\leq\frac{11}{2N-7}.$$

(Here we impose the condition "≥ 4" the ensure that the denominator is positive.) Also,

$$\frac{11}{2N-7}<\varepsilon \iff \frac{11}{\varepsilon}<2N-7 \iff N>\frac{11}{2\varepsilon}+\frac{7}{2}.$$

Then by **Archimedean Property**, pick $N\in\mathbb{N}$ such that

$$N>\max\left\{\frac{11}{2\varepsilon}+\frac{7}{2},4\right\}.$$

Hence whenever $n\geq N$, we have

$$\left|\frac{4n-3}{2n-7}-2\right|\leq\frac{11}{2N-7}<\varepsilon.$$

Exercise. Find the limit of the sequence $\left(\dfrac{18n+2}{6n-89}\right)$ and prove your assertion.

Example 5 (c.f. Example 3.1.7)**.** Show that the sequence $(0,2,0,2,...)$ is divergent.

Solution. Let x_n be the n-term of the sequence. Then

$$x_n=\begin{cases}0, & \text{if } n \text{ is odd,}\\ 2, & \text{if } n \text{ is even.}\end{cases}$$

We need to show that the sequence does not converge to any number $x\in\mathbb{R}$. i.e., $\forall x\in\mathbb{R}$, $\exists\varepsilon>0$ such that $\forall N\in\mathbb{N}$, $\exists n\geq N$ such that $|x_n-x|\geq\varepsilon$. The idea is to pick a term of the sequence at the back that is away from x.
Let $x\in\mathbb{R}$ and take $\varepsilon=1$. For any natural number N, take n to be an even number greater than N if $x\leq 1$ and take n to be an odd number greater than N if $x>1$. Then

- if $x\leq 1$, $|x_n-x|=|2-x|=2-x\geq 1=\varepsilon$.
- if $x>1$, $|x_n-x|=|0-x|=x\geq 1=\varepsilon$.

In any cases, there exists $n\geq N$ such that $|x_n-x|\geq\varepsilon$. The result follows.

Exercise. Show that the sequence $(6,8,9,6,8,9,...)$ is divergent.

Example 6 (c.f. section 3.1, Ex.14)**.** Let $b \in \mathbb{R}$ satisfies $0 < b < 1$. Show that $\lim(nb^n) = 0$.

Solution. Let $a = (1/b) - 1$. Then $a > 0$ and $b = 1/(1+a)$. Hence

$$|nb^n - 0| = nb^n = \frac{n}{(1+a)^n}.$$

By **Binomial Theorem**, if $n \geq 2$, (≥ 2 to make sure that the a^2 term exist.)

$$(1+a)^n = 1 + na + \frac{1}{2}n(n-1)a^2 + \cdots \geq \frac{1}{2}n(n-1)a^2.$$

Hence if $n \geq N \geq 2$,

$$|nb^n - 0| \leq \frac{2n}{n(n-1)a^2} = \frac{2}{(n-1)a^2} \leq \frac{2}{(N-1)a^2}.$$

With the same trick, note that

$$\frac{2}{(N-1)a^2} < \varepsilon \iff N > \frac{2}{a^2\varepsilon} + 1$$

Let $\varepsilon > 0$. By **Archimedean Property**, there exists a natural number N such that

$$N > \max\left\{\frac{2}{a^2\varepsilon} + 1, 2\right\}.$$

Hence if $n \geq N$, we have

$$|nb^n - 0| \leq \frac{2}{(N-1)a^2} < \varepsilon.$$